

ON TORSORS UNDER ABELIAN VARIETIES

A. BERTAPELLE

ABSTRACT. Let A_K be an abelian variety over a local field K of mixed characteristic and with algebraically closed residue field. We provide a geometric construction (via the relative Picard functor) of the Shafarevich duality between the group of isomorphism classes of torsors under A_K and the “fundamental group” of the Néron model of the dual abelian variety A'_K . An analogous construction works over fields of positive characteristic p providing a duality on the prime-to- p parts.

Let R be a complete discrete valuation ring with field of fractions K and algebraically closed residue field k . Let us denote by $j: \text{Spec}(K) \rightarrow \text{Spec}(R)$ the usual open immersion. Let A_K, A'_K be dual abelian varieties over K , and A, A' their Néron models.

Shafarevich’s pairing (cf. [1], [3], [2]) provides an isomorphism

$$(1) \quad H_{\mathbb{A}}^1(K, A_K) \xrightarrow{\sim} \text{Ext}(\mathbf{Gr}(A'^0), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A'^0)), \mathbb{Q}/\mathbb{Z}),$$

where A'^0 is the identity component of A' , \mathbf{Gr} denotes the perfection of the Greenberg realization functor and $\pi_1(\mathbf{Gr}(A'^0))$ is the fundamental group of the proalgebraic groups $\mathbf{Gr}(A'^0)$ (see 1.1). The existence of the pairing was proved by Shafarevich for the prime-to- p parts, by Bégueri in [1] for K of characteristic 0 and by Bester and the author, respectively in [3] and [2], for the equal positive characteristic case.

In the first section of the paper we recall the original construction of Shafarevich’s duality due to Bégueri (for K of mixed characteristic). In the second section we slightly modify Bégueri’s construction using rigidifiers. The latter construction works in any characteristic. In the third section we construct a morphism as in (1) via the relative Picard functor. We show that it always coincides with the modified Bégueri construction and hence with Shafarevich duality for K of characteristic 0. In the characteristic p case it coincides with Shafarevich duality on the prime-to- p parts. The analogous result for the p parts, although expected, is still open.

1. SHAFAREVICH’S DUALITY

1.1. Proalgebraic groups and Greenberg realization. For the theory of proalgebraic groups over the algebraically closed field k we refer to [11] and [8], II.7. For the Greenberg realization functor $\text{Gr}(-)$ we refer to [4], 9.6. For its perfection $\mathbf{Gr}(-)$ see [7] III, §4. The functor $\text{Gr}(-)$ associates to any smooth group scheme Y of finite type over R a proalgebraic group scheme $\text{Gr}(Y)$ over k (in the sense of [8]) and its perfection $\mathbf{Gr}(Y)$ is a proalgebraic group in the sense of [11].

In the category of proalgebraic groups, the component group functor π_0 admits a left derived functor π_1 which is left exact. We list below well-known facts used in this paper. Simply assuming them, the reader will be able to follow the proofs even if he/she is not familiar with the theory of proalgebraic groups.

- i) If Y is a smooth group scheme of finite type, $\pi_0(\mathbf{Gr}(Y))$ will coincide with the component group of the special fibre of Y and $\pi_1(\mathbf{Gr}(Y))$ is a profinite group.
- ii) If P is a proalgebraic group and P^0 is its identity component, then $\pi_1(P) = \pi_1(P^0)$.

- iii) A short exact sequence of smooth R -group schemes of finite type $0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow 0$ provides a long exact sequence of profinite groups (cf. [11], 10.2/1)

$$0 \rightarrow \pi_1(\mathbf{Gr}(Y_1)) \rightarrow \pi_1(\mathbf{Gr}(Y_2)) \rightarrow \pi_1(\mathbf{Gr}(Y_3)) \rightarrow \pi_0(\mathbf{Gr}(Y_1)) \rightarrow \pi_0(\mathbf{Gr}(Y_2)) \rightarrow \pi_0(\mathbf{Gr}(Y_3)) \rightarrow 0,$$

1.2. The component group of a torus. Recall that the Néron model T of a torus T_K is locally of finite type, but, in general, not of finite type over R . Let X_K denotes the group of characters of T_K . Then T is of finite type, i.e., its component group is torsion, if $X_K(K) = 0$ (cf. [4], 10.2/1).

Lemma 1. *Let $f: T_{1,K} \rightarrow T_{2,K}$ be an isogeny of tori with kernel a finite group scheme F_K . The group $H_{\text{fl}}^1(K, F_K) = T_{2,K}(K)/T_{1,K}(K)$ has a canonical proalgebraic structure.*

Proof. (Cf. [1], 4.3.) Let $X_{i,K}$ be the character group of $T_{i,K}$ and T_i the Néron model, $i = 1, 2$. Denote by $T_{1,K}^{(d)}$ the torus whose group of characters is the constant free group $X_{1,K}(K)$. Similar for $T_{2,K}^{(d)}$. They are split tori with the same component group, say \mathbb{Z}^r . Furthermore the isogeny f induces an isogeny $f^{(d)}: T_{1,K}^{(d)} \rightarrow T_{2,K}^{(d)}$ that is injective on component groups. The torus $T'_{i,K}$, kernel of the quotient map $T_{i,K} \rightarrow T_{i,K}^{(d)}$, admits a Néron model of finite type because its group of characters is $X'_{i,K} = X_{i,K}/X_{i,K}(K)$. Hence, using the exact sequences $\pi_0(T'_i) \rightarrow \pi_0(T_i) \rightarrow \pi_0(T_i^{(d)}) \rightarrow 0$, one sees that the kernel and the cokernel of the homomorphism $\pi_0(T_1) \rightarrow \pi_0(T_2)$ are finite groups. The identity components of the Néron models T_i are smooth group schemes of finite type ([12], VI A, 2.4). Hence their perfect Greenberg realizations are proalgebraic groups. Let us denote by P the cokernel of the map $\mathbf{Gr}(T_1^0) \rightarrow \mathbf{Gr}(T_2^0)$. Now, the cokernel of the map $\mathbf{Gr}(T_1) \rightarrow \mathbf{Gr}(T_2)$ is extension of the finite group $\pi_0(T_2)/\pi_0(T_1)$ by the quotient of P by a finite constant group. Hence it is proalgebraic. \square

We will use in the next sections the following result:

Lemma 2. *Let $0 \rightarrow T \rightarrow G \rightarrow A^0 \rightarrow 0$ be an exact sequence of smooth R -group schemes over R where T is the Néron model of a torus and A^0 is the identity component of the Néron model of an abelian variety. It induces a morphism $\pi_1(\mathbf{Gr}(A^0)) \rightarrow \pi_0(T)_{\text{tor}}$ where the index tor indicates the torsion subgroup.*

Proof. If T is of finite type, the proof is immediate. Indeed $\pi_0(T)_{\text{tor}} = \pi_0(T) = \pi_0(\mathbf{Gr}(T))$ is finite and, on applying the Greenberg functor to the sequence above, we get an extension

$$0 \rightarrow \mathbf{Gr}(T) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Gr}(A^0) \rightarrow 0.$$

The desired map follows then from long exact sequence of the π_i 's.

Suppose that T is locally of finite type. As k is algebraically closed, $\pi_0(T) = \pi_0(T)_{\text{tor}} \times \pi_0(T)_{\text{fr}}$. Let T^{ft} be the maximal subgroup of T such that its component group is finite, i.e., T^{ft} contains the identity component T^0 and $\pi_0(T^{\text{ft}}) = \pi_0(T)_{\text{tor}}$. The extension G is the push-out along the inclusion $T^{\text{ft}} \rightarrow T$ of a unique (up to isomorphism) extension $0 \rightarrow T^{\text{ft}} \rightarrow G^{\text{ft}} \rightarrow A^0 \rightarrow 0$ because the quotient T/T^{ft} satisfies the hypothesis in [13], §5.7, 5.5. Hence we can proceed as done above with G^{ft} in place of G obtaining a map $\pi_1(\mathbf{Gr}(A)) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(T)_{\text{tor}}$. \square

1.3. Bégueri's construction. Suppose in this section that K has characteristic 0. Given K -schemes Z_K, S_K let us denote by Z_{S_K} the fibre product $Z_K \times_K S_K$, viewed as a scheme over S_K .

Let X_K be a K -torsor under A_K and let n be a positive integer such that $n \cdot X_K$ is trivial; the order of X_K is the minimum among such integers. The torsor X_K corresponds to a n -torsion element in $H_{\text{fl}}^1(K, A_K) = \text{Ext}^1(\mathbb{Z}, A_K)$ and hence to an extension

$$(2) \quad 0 \rightarrow A_K \rightarrow B_K \rightarrow \mathbb{Z} \rightarrow 0,$$

that is the pull-back along $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ of a, not unique, extension

$$(3) \quad \eta: 0 \rightarrow A_K \xrightarrow{\alpha} E_K \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0,$$

precisely X_K is the fibre at $1 \in \mathbb{Z}/d\mathbb{Z}$. Let us denote by

$$(4) \quad \eta_n: 0 \rightarrow {}_nA_K \rightarrow {}_nE_K \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

the sequence of n -torsion subgroups.

Consider the exact sequence

$$(5) \quad 0 \rightarrow \mu_n \rightarrow V_{nE_K}^* \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} \xrightarrow{\tau} \underline{\text{Ext}}^1(E_K, \mathbb{G}_m) = A'_K \rightarrow 0$$

(cf. [1], 2.3.2) where we denote by $V_{nE_K}^*$ the Weil restriction torus ([4], 7.6)

$$\mathfrak{R}_{nE_K/K}(\mathbb{G}_{m,nE_K}) = \underline{\text{Mor}}(nE_K, \mathbb{G}_{m,K})$$

representing the functor that associates to a K -scheme S the group $\mathbb{G}_{m,K}(S \times_K nE_K)$. Observe that

$$\mu_n = \underline{\text{Hom}}(\mathbb{Z}/d\mathbb{Z}, \mathbb{G}_m) = \underline{\text{Hom}}(E_K, \mathbb{G}_m).$$

The first map in (5) maps a homomorphism $f: E_K \rightarrow \mathbb{G}_{m,K}$ to its restriction to nE_K , the second arrow sends $g \in \mathbb{G}_m(nE_K)$ to (the isomorphism class of) the trivial extension endowed with the section g , the latter map forgets the rigidification along nE_K .

We describe now Bégueri's construction of Shafarevich's duality following [1]. Let F_K be a finite K -group scheme and F_K^D its Cartier dual. There is a short exact sequence (cf. [1], 2.2.1)

$$(6) \quad 0 \rightarrow F_K^D \rightarrow V_{F_K}^* \rightarrow \underline{\text{Ext}}^1(F_K, \mathbb{G}_m)_{F_K} \rightarrow 0,$$

where the first map forgets the group structure and the second map associates to a $f \in \mathbb{G}_{m,K}(F_K)$ the trivial extension endowed with the rigidification induced by f . Finally we introduce the following exact sequence (cf. [1], 2.3.1)

$$(7) \quad 0 \rightarrow V_{nA_K}^* \rightarrow \underline{\text{Ext}}^1(A_K, \mathbb{G}_m)_{nA_K} \xrightarrow{\tau} A'_K \rightarrow 0.$$

At first Bégueri constructs in [1], 8.2.2, a map

$$(8) \quad \Gamma: H_{\text{fl}}^1(K, {}_nA_K) \rightarrow \text{Ext}^1(\mathbf{Gr}(A'), \underline{H}^1(K, \mu_n))$$

as follows: any element in $H_{\text{fl}}^1(K, {}_nA_K)$ corresponds to a sequence η_n as in (4). Consider now the diagram

$$(9) \quad \begin{array}{ccccc} \mu_n & \xlongequal{\quad} & \mu_n & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ V_{\mathbb{Z}/n\mathbb{Z}}^* & \xrightarrow{\quad} & V_{nE_K}^* & \xrightarrow{\quad} & V_{nA_K}^* \\ \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 \\ \underline{\text{Ext}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)_{\mathbb{Z}/n\mathbb{Z}} & \longrightarrow & \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} & \longrightarrow & \underline{\text{Ext}}^1(A_K, \mathbb{G}_m)_{nA_K} \\ \downarrow & & \downarrow \tau & & \downarrow \\ 0 & & A'_K & \xlongequal{\quad} & A'_K \end{array}$$

where the rows are complexes and the vertical sequences are those in (6), for $F_K = \mathbb{Z}/n\mathbb{Z}$, (5), (7), respectively. Since K has characteristic 0, the second row consists of tori, the third row consists of semi-abelian varieties. Hence they all admit Néron models. On passing to the perfection of the Greenberg realization

of the Néron models and considering the cokernels of the maps induced by v_1, v_2, v_3 , one gets a complex of proalgebraic groups¹

$$(10) \quad 0 \rightarrow \underline{\underline{H}}^1(K, \mu_n) \rightarrow \underline{\underline{\text{Ext}}}^1(E_K, \mathbb{G}_m) \rightarrow \mathbf{Gr}(A') \rightarrow 0;$$

it is indeed an exact sequence because on k -points it induces the exact sequence

$$0 \rightarrow H_{\text{fl}}^1(K, \mu_n) = \text{Ext}^1(\mathbb{Z}/d\mathbb{Z}, \mathbb{G}_m) \rightarrow \text{Ext}^1(E_K, \mathbb{G}_m) \rightarrow A'(R) = \text{Ext}^1(A_K, \mathbb{G}_m) \rightarrow 0.$$

We have then associated to (4) an extension of $\mathbf{Gr}(A')$ by $\underline{\underline{H}}^1(K, \mu_n)$: this is the image of (4) via Γ .

The homomorphism

$$(11) \quad \psi_n: H_{\text{fl}}^1(K, {}_nA_K) \longrightarrow \text{Ext}^1(\mathbf{Gr}(A'^0), \mathbb{Z}/n\mathbb{Z})$$

in [1], 8.2.3, is then obtained by applying first Γ , then the pull-back along $\mathbf{Gr}(A'^0) \rightarrow \mathbf{Gr}(A')$ and, finally, the push-out along $\underline{\underline{H}}^1(K, \mu_n) \rightarrow \pi_0(\underline{\underline{H}}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z}$.

Let us denote by

$$(12) \quad \psi_n(\eta_n): 0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow W(X_K) \rightarrow \mathbf{Gr}(A'^0) \rightarrow 0$$

the image of (4) via ψ_n ,

Recall now that (cf. [11], 5.4)

$$\text{Ext}(\mathbf{Gr}(A'^0), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A'^0)), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}).$$

In terms of homomorphisms of profinite groups, the extension (12) corresponds to a map

$$(13) \quad u^\tau = u_{X_K}^\tau: \pi_1(\mathbf{Gr}(A'^0)) \longrightarrow \pi_0(\underline{\underline{H}}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

deduced from (10) (or the same, from the pull-back of (10) along $\mathbf{Gr}(A'^0) \rightarrow \mathbf{Gr}(A')$ via the long exact sequence of π_i 's).

It is proved in [1], 8.2.3 that: a) the extension $\psi_n(\eta_n)$ in (12) depends only on the sequence (2), i.e., on the torsor X_K , b) its formation behaves well w.r.t. the inclusions ${}_nH_{\text{fl}}^1(K, {}_nA_K) \rightarrow {}_{n'}H_{\text{fl}}^1(K, {}_{n'}A_K)$, and $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n'\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ for $n|n'$, and c) it provides the duality in (1). Hence:

Shafarevich's duality in (1) maps the torsor X_K to the homomorphism $u_{X_K}^\tau$ in (13).

1.3.1. *An alternative construction of $\psi_n(\eta_n)$ in (12) (in view of further applications).* The kernel of τ in (7) is a torus, that for brevity we denote by T_K^τ . Let T^τ be its Néron model. We have an exact sequence

$$0 \rightarrow T_K^\tau \rightarrow \underline{\underline{\text{Ext}}}^1(E_K, \mathbb{G}_m)_{nE_K} \xrightarrow{\tau_K} A'_K \rightarrow 0.$$

which extends to an exact sequence of Néron models

$$(14) \quad 0 \rightarrow T^\tau \rightarrow j_*\underline{\underline{\text{Ext}}}^1(E_K, \mathbb{G}_m)_{nE_K} \xrightarrow{\tau} A' \rightarrow 0.$$

On applying the perfection of the Greenberg functor we get an exact sequence

$$(15) \quad 0 \rightarrow \mathbf{Gr}(T^\tau) \rightarrow \mathbf{Gr}(j_*\underline{\underline{\text{Ext}}}^1(E_K, \mathbb{G}_m)_{nE_K}) \xrightarrow{\tau} \mathbf{Gr}(A') \rightarrow 0$$

where the first two groups are not proalgebraic, in general. Nevertheless, on applying the Greenberg functor to the morphism of Néron models $j_*V_{nE_K}^* \rightarrow T^\tau$ we get a homomorphism whose cokernel is a proalgebraic group (cf. Lemma 1) and whose group of k -points is $H^1(K, \mu_n)$; we will write

$$(16) \quad \mathbf{Gr}(j_*V_{nE_K}^*) \rightarrow \mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \underline{\underline{H}}^1(K, \mu_n) \rightarrow 0.$$

¹Note that one applies here Lemma 1 to the isogeny v_1 .

Now take the push-out of (15) along h^τ ; by construction, the resulting exact sequence is the one in (10), i.e., the image of (4) via Γ . Hence, if one considers the pull-back of (15) along $\mathbf{Gr}(A^0) \rightarrow \mathbf{Gr}(A')$,

$$(17) \quad 0 \rightarrow \mathbf{Gr}(T^\tau) \rightarrow U \rightarrow \mathbf{Gr}(A^0) \rightarrow 0,$$

and then the push-out of (17) along the composition of maps

$$\mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \underline{\mathbf{H}}^1(K, \mu_n) \rightarrow \pi_0(\underline{\mathbf{H}}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z},$$

one gets the extension $\psi_n(\eta_n)$ in (12), i.e., the image of X_K via Shafarevich's duality

Thanks to this new description of Shafarevich's map, we can characterize of the map u^τ in (13) as follows:

First consider the pull-back of (14) along $A^0 \rightarrow A'$. As we have seen in the proof of Lemma 2, this extension is the push-out of an extension

$$0 \rightarrow T^{\tau, \text{ft}} \rightarrow G \rightarrow A^0 \rightarrow 0$$

where $T^{\tau, \text{ft}}$ is the maximal subgroup scheme of finite type of T^τ . The pull-back of the sequence in (15) along $\mathbf{Gr}(A^0) \rightarrow \mathbf{Gr}(A')$ is then isomorphic to the push-out of

$$(18) \quad 0 \rightarrow \mathbf{Gr}(T^{\tau, \text{ft}}) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Gr}(A^0) \rightarrow 0$$

along the composition of maps $h^{\tau, \text{ft}}: \mathbf{Gr}(T^{\tau, \text{ft}}) \rightarrow \mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \underline{\mathbf{H}}^1(K, \mu_n)$. Hence

$$(19) \quad \boxed{u_{X_K}^\tau = \pi_0(h^{\tau, \text{ft}}) \circ w}$$

where the homomorphism $w: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\tau, \text{ft}}))$ is deduced from (18) via the long exact sequence of π_i 's.

2. AN ALTERNATIVE CONSTRUCTION USING RIGIDIFIATORS

Let X_K be a torsor under an abelian variety A_K . We will see in this section how the homomorphism u^τ in (13) (and in (19)) can be constructed using a rigidificator x_K of the relative Picard functor $\text{Pic}_{X_K/K}$.

2.1. Rigidificators. A rigidificator x_K of the relative Picard functor $\text{Pic}_{X_K/K}$ is a finite closed subscheme x_K of X_K such that for any K -scheme S_K the map

$$\Gamma(X_{S_K}, \mathcal{O}_{X_{S_K}}) \rightarrow \Gamma(x_{S_K}, \mathcal{O}_{x_K \times_K S_K})$$

is injective. (cf. [9], 2.1.1).

Lemma 3. *Let X_K be a torsor under A_K of order d . Let d' be the separable index of X_K , i.e., the greatest common divisor of the degrees of its finite separable splitting extensions. Then $d|d'$ and they have the same prime factors. If A_K is an elliptic curve, then $d = d'$ and the index is indeed the degree of a minimal separable splitting extension.*

Proof. One can invoke [5], Proposition 5. Alternatively one proves, using the restriction and corestriction maps, that $n \cdot X_K = 0$ if X_K becomes trivial over a finite separable extension K'/K of degree n . Hence $d|n$. Suppose now given separable extensions $K \subseteq L \subseteq L'$ with $(d, [L' : L]) = 1$ and $X_{L'} = 0$. Then $[L' : L] \cdot X_L = 0$ in $H_{\mathbb{A}}^1(L, A_L)$. However the order of X_L in $H_{\mathbb{A}}^1(L, A_L)$ divides d and thus $X_L = 0$. Hence d, d' have the same prime factors.

For the latter assertion on elliptic curves see [5], p. 670, or [6], Theorems 1 & 4, or [10], 2.1.2. □

Remark 4. Let $x_K = \text{Spec}(K')$ with K'/K a finite separable extension of degree n . Then the torus $V_{x_K}^*$ is the Weil restriction $\mathfrak{R}_{K'/K}(\mathbb{G}_{m,K'})$, it has component group isomorphic to \mathbb{Z} and the closed immersion $\mathbb{G}_{m,K} \rightarrow V_{x_K}^*$ (i.e., the inclusion $K^* \subset K'^*$ on K -sections) induces the n -multiplication $n: \mathbb{Z} \rightarrow \mathbb{Z}$ on component groups of Néron models.

2.2. The alternative construction of Shafarevich's duality. The main idea is to use a rigidificator x_K in place of ${}_n E_K$ in (5). The advantage is that the new construction works even for K of positive characteristic; in this case we choose x_K étale so that $V_{x_K}^* = \mathfrak{R}_{x_K/K}(\mathbb{G}_{m,x_K})$ is still a torus.

Observe that a rigidificator x_K is a closed subscheme of E_K and the homomorphism

$$(20) \quad \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{x_K}^*$$

is still a closed immersion. Indeed any homomorphism $f: E_K \rightarrow \mathbb{G}_m$ factors through $\rho: E_K \rightarrow \mathbb{Z}/d\mathbb{Z}$ and if $f|_{x_K} = 0$ then $f|_{X_K} = 0$ because x_K is a rigidificator. However X_K is the fibre at 1 of ρ and hence also $f = 0$. We then have an exact sequence

$$(21) \quad 0 \rightarrow \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{x_K}^* \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} \rightarrow A'_K \rightarrow 0.$$

More generally, we will say that a finite étale subscheme Z_K of E_K satisfies property $(*)$ if

$$(*) \quad \text{the canonical map } \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{Z_K}^* \text{ is a closed immersion.}$$

For any such étale subscheme x_K we can construct an exact sequence as in (21).

Denote by T_K^x the torus $V_{x_K}^*/\mu_n$ and omit the exponent x if the rigidificator x_K is fixed. The sequence (21) induces an exact sequence

$$(22) \quad 0 \rightarrow T_K \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} \rightarrow A'_K \rightarrow 0,$$

and hence a n exact sequence (cf. Lemma 2/proof)

$$(23) \quad 0 \rightarrow T^{\text{ft}} \rightarrow G_1 \rightarrow A'^0 \rightarrow 0,$$

where T^{ft} is the maximal subgroup of finite type of the Néron model T of T_K . Consider now the cokernel

$$(24) \quad \text{Gr}(j_* V_{x_K}^*) \xrightarrow{g^x} \text{Gr}(T) \xrightarrow{h} \underline{\text{H}}^1(K, \mu_n) \rightarrow 0$$

of the homomorphism between the Greenberg realizations of the Néron models of $V_{x_K}^*$ and T_K ; by Lemma 1 it is a proalgebraic group whose group of k -points is $\text{H}_{\mathbb{A}}^1(K, \mu_n)$.

Lemma 5. The proalgebraic group $\underline{\text{H}}^1(K, \mu_n)$ in (24) does not depend on the étale finite subscheme x_K chosen to construct it. In particular it coincides with the one in (5).

Proof. Let $x_K \subset y_K$ be finite étale subschemes of E_K satisfying $(*)$. We have canonical morphisms $f^V: V_{y_K}^* \rightarrow V_{x_K}^*$, $f^T: T_K^y \rightarrow T_K^x$, such that $f^T \circ g^y = g^x \circ f^V$. Hence the proalgebraic group constructed in (24) for x_K is canonically isomorphic to the one constructed via y_K . \square

In order to provide a more useful description of the map u^τ in (13), consider the perfect Greenberg realization of (23)

$$(25) \quad 0 \rightarrow \text{Gr}(T^{\text{ft}}) \rightarrow \text{Gr}(G_1) \rightarrow \text{Gr}(A'^0) \rightarrow 0,$$

and then its push-out along the composition of maps

$$(26) \quad h^{\text{ft}}: \text{Gr}(T^{\text{ft}}) \rightarrow \text{Gr}(T) \xrightarrow{h} \underline{\text{H}}^1(K, \mu_n).$$

We obtain an exact sequence

$$(27) \quad \zeta: 0 \rightarrow \underline{\mathbf{H}}^1(K, \mu_n) \rightarrow W' \rightarrow \mathbf{Gr}(A'^0) \rightarrow 0$$

and hence a homomorphism

$$u_{X_K} = u: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\underline{\mathbf{H}}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

such that

$$(28) \quad u = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}},$$

where $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(T)_{\text{tor}}$ is deduced from the long exact sequence of π_i 's of (25).

Proposition 6. *The association $X_K \mapsto u_{X_K}$ provides a homomorphism*

$$\mathbf{H}^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}).$$

If $\text{char}(K) = 0$ it is Shafarevich duality in (1), i.e., $u_{X_K} = u_{X_K}^\tau$.

Proof. We start showing that the construction of $u: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$ in (28) does not depend on the choices of x_K , n and $\eta \in \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K)$ above X_K .

At first we see that u does not depend on the étale finite closed subscheme x_K of E_K satisfying (*). Let $x_K \subset y_K$ be two étale subschemes of E_K satisfying (*). Denote by $T_K^x, h^x, h^{\text{ft},x}, u'^x, u^x$ respectively the torus in (22), the maps in (24), (26) and (28) for x_K , and similarly for y_K . We have a canonical morphism of tori $T_K^y \rightarrow T_K^x$ and it induces a morphism $\beta: T_K^{y,\text{ft}} \rightarrow T_K^{x,\text{ft}}$ between the maximal subgroups of finite type. Denote by $\beta': \mathbf{Gr}(T^{y,\text{ft}}) \rightarrow \mathbf{Gr}(T^{x,\text{ft}})$ the corresponding map on Greenberg realizations of Néron models. It holds $\beta' \circ h^{x,\text{ft}} = h^{y,\text{ft}}$. One has $\pi_0(h^{y,\text{ft}}) = \pi_0(h^{x,\text{ft}}) \circ \pi_0(\beta')$. Furthermore the sequence (25) for x_K is the push-out along β' of the sequence (25) for y_K . Hence $u^{x,\text{ft}} = \pi_0(\beta') \circ u^{y,\text{ft}}$. We conclude then that

$$(29) \quad u^x = \pi_0(h^{x,\text{ft}}) \circ u^{x,\text{ft}} = \pi_0(h^{x,\text{ft}}) \circ \pi_0(\beta') \circ u^{y,\text{ft}} = \pi_0(h^{y,\text{ft}}) \circ u^{y,\text{ft}} = u^y.$$

Let now n, \hat{n} be positive integers such that $n \cdot X_K = 0$ and $n|\hat{n}$. We can consider the pull-back $\hat{\eta}$ of η in (3) along the projection $\mathbb{Z}/\hat{n}\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. If we proceed with $\hat{\eta}$ as we have done for η , we get a map $\hat{u}: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$. Observe that the 2-fold extension (21) for $\hat{\eta}$ is the push-out along $\mu_n \rightarrow \mu_{\hat{n}}$ of (21) and that the map $\pi_0(\underline{\mathbf{H}}^1(K, \mu_n)) \rightarrow \pi_0(\underline{\mathbf{H}}^1(K, \mu_{\hat{n}}))$ is the inclusion $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/\hat{n}\mathbb{Z}$. It is now immediate to check that the maps \hat{u} and u coincide.

We have then obtained a map

$$(30) \quad \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}), \quad \eta \mapsto u.$$

To check that it is indeed an homomorphism, observe that it is functorial in A_K . Furthermore we could repeat the construction with any finite constant group F_K in place of $\mathbb{Z}/n\mathbb{Z}$ obtaining in this way a map

$$\text{Ext}^1(F_K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \pi_0(\underline{\mathbf{H}}^1(K, F_K^D)))$$

with F_K^D the Cartier dual of F_K . This construction is functorial in F_K . The functoriality results is sufficient to conclude that the map in (30) is a homomorphism, because the Baer's sum of two extensions as in (3) is done first by taking the direct sum of the two extensions, then by applying the push-out along the multiplication of A'_K and finally by applying the pull-back along the diagonal $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Suppose now n and x_K fixed. We show now that the map u is trivial if X_K is trivial, i.e., the map in (30) factors through $\mathbf{H}_\text{fl}^1(K, A_K)$. Suppose X_K is trivial and choose a K -point x_K of X_K . In particular, $V_{x_K}^* = \mathbb{G}_{m,K}$, $T_K = \mathbb{G}_{m,K}$ and $\pi_0(T) = \mathbb{Z}$. Hence $T^{\text{ft}} = \mathbb{G}_{m,R}$, the homomorphism $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = 0$ is the zero map and $u = 0$.

Suppose now $\text{char}(K) = 0$. To see that the homomorphism $X_K \mapsto u_{X_K}$ is Shafarevich's duality, it is sufficient to check that the homomorphisms u^τ in (13) and u in (28) coincide. Consider then a finite separable extension K'/K splitting (4) and a point $x_K = \text{Spec}(K')$ of ${}_nE_K$ above 1. It is a rigidificator of $\text{Pic}_{X_K/K}$. Pose $y_K = {}_nE_K$. Then, using notations as above, u^y coincides with the map u^τ in (19) and one can repeat the arguments in (29). \square

Remark 7. *The original construction by B  gueri works only for K of characteristic zero because in the case of positive characteristic the scheme $V_{{}_nE_K}^*$ (and hence T_K^τ) might not be a torus; in particular it might not admit a N  ron model. The construction via rigidificators described in this section works in any characteristic. For $\text{char}(K) = p$ it is not clear that it provides Shafarevich duality. We will see in Proposition 9 that this is the case on the prime-to- p parts.*

3. A CONSTRUCTION VIA THE PICARD FUNCTOR

In this section we present a third possible construction of a homomorphism as in (1) which makes use the relative Picard functor. We will see that it always coincide with the one in Proposition 6 and hence with Shafarevich's duality in the characteristic 0 case.

Let X_K be a torsor under A_K and $x_K = \text{Spec}(K')$ a rigidificator of $\text{Pic}_{X_K/K}$ with K'/K a finite separable extension. It exists by Lemma 3. No assumption on the characteristic of K is made.

Consider the usual exact sequence (cf. [9], 2.4.1)

$$(31) \quad 0 \rightarrow V_{X_K}^* \rightarrow V_{x_K}^* \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \rightarrow A'_K \rightarrow 0.$$

Observe that $V_{X_K}^* = \mathfrak{R}_{X_K/K}(\mathbb{G}_{m, X_K}) = \mathbb{G}_{m, K}$ ([9], 2.4.3), $V_{x_K}^*$ is a torus and hence so too is $N_K := V_{x_K}^*/\mathbb{G}_{m, K}$. Denote by N its N  ron model. Observe that it follows from Remark 4 that the component group of N is cyclic of order n , hence the perfection of its Greenberg realization is a proalgebraic group.

We proceed as in the previous section, first by passing to N  ron models and then applying the Greenberg realization to the sequence

$$(32) \quad 0 \rightarrow N_K \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \xrightarrow{h_K} A'_K \rightarrow 0$$

so that we obtain an exact sequence of proalgebraic groups

$$(33) \quad 0 \rightarrow \mathbf{Gr}(N) \rightarrow \mathbf{Gr}(j_*(\text{Pic}_{X_K/K}, x_K)^0) \xrightarrow{h} \mathbf{Gr}(A') \rightarrow 0$$

and hence a homomorphism

$$(34) \quad v = v_{X_K} : \pi_1(\mathbf{Gr}(A')) \longrightarrow \pi_0(\mathbf{Gr}(N)) = \mathbb{Z}/n\mathbb{Z}.$$

In order to compare this construction with the (modified) B  gueri construction of the previous section, i.e., in order to compare the maps u (28) and v in (34), we consider the following diagram

$$(35) \quad \begin{array}{ccccccc} 0 & \longrightarrow & [V_{x_K}^*/\mu_n] = T_K & \longrightarrow & \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A'_K \longrightarrow 0 \\ & & \downarrow t_K & & \downarrow f_K & & \parallel \\ 0 & \longrightarrow & [V_{x_K}^*/V_{X_K}^*] = N_K & \longrightarrow & (\text{Pic}_{X_K/K}, x_K)^0 & \xrightarrow{h_K} & A'_K \longrightarrow 0 \end{array}$$

where the upper sequence is (22), the lower one is (32) and f_K associates to a \mathbb{G}_m -extension L_K of E_K endowed with a x_K -section σ its restriction (as torsor) to X_K endowed with the trivialization along x_K induced by σ . The morphism t_K is surjective and its kernel is $\mathbb{G}_{m, K} = V_{X_K}^*/\mu_n = \mathbb{G}_{m, K}/\mu_n$.

Consider now the induced diagram on Néron models.

$$\begin{array}{ccccccc}
0 & \longrightarrow & T^{\text{ft}} & \longrightarrow & G_1 & \longrightarrow & A'^0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & t^{\text{ft}} \curvearrowright T & \longrightarrow & j_* \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A' \longrightarrow 0 \\
& & \downarrow & & \downarrow f & & \parallel \\
0 & \longrightarrow & N & \longrightarrow & j_*(\text{Pic}_{X_K/K}, x_K)^0 & \longrightarrow & A' \longrightarrow 0
\end{array}$$

where the first row is (23). The homomorphism u in (28) is the composition of the homomorphism $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A'^0)) \longrightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}}))$ (deduced from the upper exact sequence) with the homomorphism

$$\pi_0(h^{\text{ft}}): \pi_0(T^{\text{ft}}) = \pi_0(\mathbf{Gr}(T^{\text{ft}})) \rightarrow \pi_0(\underline{\mathbb{H}}^1(K, \mu_n)).$$

Now it follows from the above diagram that the map $v: \pi_1(\mathbf{Gr}(A')) \longrightarrow \pi_0(\mathbf{Gr}(N))$ in (34), obtained from the lower exact sequence, satisfies

$$(36) \quad v = \pi_0(t^{\text{ft}}) \circ u^{\text{ft}}.$$

In order to explicate the relation between u and v , consider the following diagram

$$\begin{array}{ccccccc}
& & & & \mathbb{G}_{m,K} & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & \mu_n & \longrightarrow & V_{x_K}^* & \longrightarrow & T_K \longrightarrow 0 \\
& & \downarrow & & \parallel & & \downarrow t_K \\
0 & \longrightarrow & V_{X_K}^* = \mathbb{G}_{m,K} & \longrightarrow & V_{x_K}^* & \longrightarrow & N_K \longrightarrow 0 \\
& & \downarrow n & & & & \\
& & \mathbb{G}_{m,K} & & & &
\end{array}$$

and consider the induced diagram on component groups of Néron models

$$\begin{array}{ccccccc}
& & & & \mathbb{Z} & & \pi_0(T^{\text{ft}}) = \pi_0(T)_{\text{tor}} \\
& & & & \downarrow & & \swarrow \iota \\
& & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(T) & & \\
& & \parallel & & \downarrow & & \swarrow \pi_0(t^{\text{ft}}) \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(N) \longrightarrow 0
\end{array}$$

where ι is the inclusion map and the vertical sequence is left exact because \mathbb{Z} is torsion free (cf. [13], VIII 5.5).

We insert this diagram into a bigger diagram

$$(37) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(T) & \xrightarrow{q_1} & \pi_0(\underline{H}^1(K, \mu_n)) \longrightarrow 0 \\ & & \downarrow & & \downarrow q_2 & \nearrow \iota & \uparrow \pi_0(h^{\text{ft}}) \\ & & \pi_0(N) & \xlongequal{\quad} & \pi_0(N) & \xleftarrow{\pi_0(t^{\text{ft}})} & \pi_0(T^{\text{ft}}) \end{array}$$

where $q_1 \circ \iota = \pi_0(h^{\text{ft}})$ and $q_2 \circ \iota = \pi_0(t^{\text{ft}})$. By Remark 4, $\pi_0(N) = \mathbb{Z}/n\mathbb{Z}$ and the vertical sequence on the left coincides with the upper horizontal sequence. Hence the vertical sequence in the middle splits as well as the horizontal sequence in the middle. The identifications $\pi_0(V_{x_K}^*) = \mathbb{Z}$ induces then the identification

$$\pi_0(N) \cong \mathbb{Z}/n\mathbb{Z} \cong \pi_0(\underline{H}^1(K, \mu_n))$$

where the first isomorphism maps the image of the class of a uniformizer $\pi' \in K'^* = V_{x_K}^*(K)$ to the class of 1, while the second isomorphism maps the class of 1 to the image of the cohomology class corresponding to a uniformizer $\pi \in K^* = \mathbb{G}_{m,K}(K)$.

Let σ be a section of q_2 . It holds $q_1 \circ \sigma = \text{id}_{\mathbb{Z}/n\mathbb{Z}}$. Furthermore $\sigma \circ q_2 \circ \iota = \iota$ because $\sigma \circ q_2 \circ \iota - \iota$ factors through \mathbb{Z} and thus is trivial because $\pi_0(T^{\text{ft}})$ is torsion. Hence

$$\pi_0(h^{\text{ft}}) = q_1 \circ \iota = q_1 \circ \sigma \circ q_2 \circ \iota = q_2 \circ \iota = \pi_0(t^{\text{ft}})$$

and hence thanks to (36) and (28), we get

$$v = \pi_0(t^{\text{ft}}) \circ u^{\text{ft}} = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}} = u.$$

We can then state the main result as follows:

Theorem 8. *Let A_K be an abelian variety over K . Then the homomorphism*

$$H^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z})$$

mapping the torsor X_K to the homomorphism

$$u_{X_K}: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$$

in (28) coincides with the homomorphism mapping X_K to the homomorphism v_{X_K} in (34). If furthermore the characteristic of K is zero, then both constructions coincide with B  gueri's construction in (13), i.e., they explicate Shafarevich duality.

We expect that the construction of (34) via relative Picard functors can be related with that in [10].

3.1. Equal characteristic case. For K of characteristic p , it is not clear either that the homomorphisms in (28), (34) are still isomorphisms or that they provide Shafarevich's duality (cf. [3] for the good reduction case and in [2] in general). However, we have a partial result on the prime-to- p parts where Shafarevich's duality is quite easy to describe.

3.1.1. *Shafarevich's duality on the prime-to- p parts.* Let $n = l^r$ be a positive integer, prime to p , and large enough to kill the l -primary parts of the component groups of A_K and A'_K . Consider the perfect cup product pairing

$$\langle \cdot, \cdot \rangle: H^1(K, {}_n A_K) \times {}_n A'_K(K) \rightarrow H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

on the (étale or flat) cohomology groups of the n -torsion points of A_K and A'_K . Given an extension η_n as in (4) (which corresponds to the torsor X_K) and a point $a \in {}_n A'_K(K)$; then $\langle \eta_n, a \rangle$ is the class of the pull-back along $a: \mathbb{Z} \rightarrow {}_n A'_K$ of the Cartier dual of η_n ,

$$\eta_n^D: 0 \rightarrow \mu_n \rightarrow {}_n E_K^D \rightarrow {}_n A'_K \rightarrow 0,$$

and it corresponds to the image of a along the boundary map $\partial: {}_n A'_K(K) \rightarrow H^1(K, \mu_n)$. Furthermore, if ${}_n A'^0$ denotes the quasi-finite subgroup of n -torsion sections of A'^0 , we have

$$\begin{aligned} \pi_1(\mathbf{Gr}(A'))/n \cdot \pi_1(\mathbf{Gr}(A')) &= {}_n A'^0(R) = {}_n A'(R)/{}_n A'(R), \\ {}_n H^1(K, A_K) &= H^1(K, {}_n A_K)/H^1(K, {}_n A_K) \end{aligned}$$

(cf. [2] §1) and Shafarevich duality on the d -primary parts

$$(38) \quad {}_n H^1(K, A_K) \times \pi_1(\mathbf{Gr}(A'))/n \cdot \pi_1(\mathbf{Gr}(A')) \rightarrow H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

is induced by the above cup product.

The map $u^\tau: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\underline{H}^1(K, \mu_n)) = H^1(K, \mu_n)$ associated to the torsor X_K can also be viewed as the composition

$$(39) \quad \pi_1(\mathbf{Gr}(A')) \xrightarrow{\delta} {}_n A'(R) = {}_n A'_K(K) \xrightarrow{\partial} H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

where the first map is deduced from the exact sequence

$$0 \rightarrow {}_n A'_K \rightarrow A'_K \xrightarrow{n} A'_K \rightarrow 0$$

on passing to Néron models. More precisely we have

$$0 \rightarrow {}_n A' \rightarrow A' \xrightarrow{n} {}_n A' \rightarrow 0$$

where ${}_n A'$ is a subgroup scheme of A' that contains \tilde{A}'^0 . In particular, on applying the perfection of the Greenberg realization functor we get a homomorphism

$$(40) \quad \pi_1(\mathbf{Gr}(A')) = \pi_1(\mathbf{Gr}({}_n A')) \rightarrow \pi_0({}_n A') = {}_n A'(R).$$

3.1.2. *Comparison result.* Let X_K be a torsor under A_K of order d with d a power of a prime integer l , $l \neq p$. Let $n = l^r$ be a multiple of d large enough to kill the l -primary parts of the component groups of A_K and A'_K . Fix an extension as (3) corresponding to X_K and let $x_K = \text{Spec}(K')$ be a rigidificator of $\text{Pic}_{X_K/K}$ contained in ${}_n E_K$. We show that the composition of the maps in (39) coincides with the map u in (28). This is sufficient to conclude that our construction via rigidificators (or equivalently via the relative Picard functor) is Shafarevich's duality on the prime-to- p parts.

With notations as in (3), observe that the n -multiplication on A_K factors through E_K so that we have a homomorphism $\gamma: E_K \rightarrow A_K$, with kernel ${}_n E_K$ such that $\gamma \circ \alpha = n$. Consider the sequence in (21). We have a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_n & \longrightarrow & {}_n E_K^D & \longrightarrow & \underline{\text{Ext}}^1(A_K, \mathbb{G}_m) & \xrightarrow{n} & A'_K & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow \gamma^* & & \parallel & & \\ 0 & \longrightarrow & \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) & \longrightarrow & V_{x_K}^* & \longrightarrow & \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A'_K & \longrightarrow & 0. \end{array}$$

Indeed ${}_nE_K^D = \underline{\mathbf{Hom}}({}_nE_K, \mathbb{G}_m)$ maps canonically to $V_{x_K}^* = \underline{\mathbf{Mor}}(x_K, \mathbb{G}_{m,K})$; hence ${}_nA'_K$ maps to the torus $T_K = V_{x_K}^*/\mu_n$ in (22). The push-out of the exact sequence $0 \rightarrow {}_nA'_K \rightarrow A'_K \rightarrow A'_K \rightarrow 0$ along ${}_nA'_K \rightarrow T_K$ provides the sequence (22) and the homomorphism γ^* sends a \mathbb{G}_m -extension of A_K to its pull-back along γ endowed with its canonical trivialization along x_K , induced by the canonical trivialization along ${}_nE_K$.

Moreover, the boundary map $\partial: {}_nA'_K(K) \rightarrow H^1(K, \mu_n)$ (of finite groups) is the composition of $\nu: {}_nA'_K(K) \rightarrow T_K(K)$ with the boundary map $h: T_K(K) \rightarrow H^1(K, \mu_n)$, i.e.

$$\partial = h \circ \nu.$$

Recall furthermore that the kernel of the n -multiplication on A' is a quasi-finite group scheme over R whose finite part is an étale finite group scheme over R of order prime to p , hence constant, because R is strictly Henselian.

On the level of proalgebraic groups we then have a diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_nA'(R) & \longrightarrow & \mathbf{Gr}(A') & \longrightarrow & \mathbf{Gr}({}_nA') \longrightarrow 0 \\ & & \downarrow \nu & & \downarrow \alpha^* & & \downarrow \\ 0 & \longrightarrow & \mathbf{Gr}(T) & \longrightarrow & \mathbf{Gr}(j_* \underline{\mathbf{Ext}}^1(E_K, \mathbb{G}_m)_{x_K}) & \longrightarrow & \mathbf{Gr}(A') \longrightarrow 0 \end{array}$$

Since the vertical map on the left factors through a map $\nu^{\text{ft}}: {}_nA'(R) \rightarrow \mathbf{Gr}(T^{\text{ft}})$, the homomorphism $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(\mathbf{Gr}(T))_{\text{tor}}$ in (28) factors through the map $\delta: \pi_1(\mathbf{Gr}(A')) \rightarrow {}_nA'(R)$ in (40) and hence

$$u^\tau = \partial \circ \delta = \pi_0(h^{\text{ft}}) \circ \pi_0(\nu^{\text{ft}}) \circ \delta = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}} = u,$$

i.e., the homomorphism $u: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Z}/n\mathbb{Z}$ in (28) coincides with that in (39). We conclude then

Proposition 9. *For any local field K with algebraically closed residue field, Shafarevich pairing coincides with the pairing constructed in Section 2, on the prime-to- p parts.*

The comparison for the p parts is still open.

Acknowledgements: We thank M. Raynaud for some precious suggestions and J. Tong for pointing out some mistakes in the first draft. We thank Progetto di Eccellenza Cariparo 2008-2009 “Differential methods in Arithmetics, Geometry and Algebra” for a financial support.

REFERENCES

- [1] L. Bégueri: Dualité sur un corps local à corps résiduel algébriquement clos. Mém. Soc. Math. Fr. (S.N.), 4, (1980)
- [2] A. Bertapelle: Local flat duality of abelian varieties. Man. math. 111 (2003) 141–161
- [3] M. Bester: Local flat duality of Abelian Varieties. Math. Ann. 235, 149–174 (1978)
- [4] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete, 21. Springer-Verlag, Berlin, 1990
- [5] S. Lang, J. Tate: Principal Homogeneous Spaces Over Abelian Varieties. Amer. J. Math. 80 (1958), 659–684
- [6] S. Lichtenbaum: The Period-Index Problem for Elliptic Curves. Amer. J. Math. 90 (1968), 1209–1223
- [7] J. Milne: Arithmetic Duality Theorems. Perspectives in Mathematics, No. 1, Academic Press, 1986
- [8] F. Oort: Commutative group schemes. Lecture Notes in Mathematics, 15 Springer-Verlag, Berlin-New York 1966
- [9] M. Raynaud: Spécialisation du foncteur de Picard. Inst. Hautes Études Sci. Publ. Math. No. 38 (1970), 27–76
- [10] J. Tong: Etude locale des torseurs sous une courbe elliptique. Preprint 2010 arXiv:1005.0462v1 [math.AG]
- [11] J.-P. Serre: Groupes proalgébriques. Inst. Hautes Études Sci. Publ. Math. No. 7, (1960)
- [12] Schémas en groupes. I: Propriétés générales des schémas en groupes. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3I). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Springer-Verlag, 1970
- [13] Groupes de monodromie en géométrie algébrique I. Séminaire de Géométrie Algébrique du Bois Marie 1967/1969 (SGA 7I) Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Mathematics 288, Springer-Verlag, 1972